A Brief History of Decision Theory

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1 From Expected Value to Expected Utility Maximization

The history of decision theory is deeply interwoven with the history of probability theory (Glimcher, 2004, p. 178) since it is concerned with choice situations where the outcomes are uncertain (Parmigiani and Inoue, 2009, p. 33). Historians of ideas typically locate the origin of probability theory around the middle of the 17th century, when a habitual gambler sought the help of French mathematician Blaise Pascal in determining the fair split of stakes in an interrupted game of chance. The ensuing exchange of ideas between Blaise Pascale and Pierre de Fermat is widely considered the hour of birth of probability theory. In the so-called "problem of points", two players contribute equal amounts to the price pot. Whoever wins the most out of a finite number of rounds, where for every round the winning chance is 50%, takes home the money. However, the game is interrupted, and the players must now decide how to split the pot[1]. A previous solution had derived the fair split based on the results of the rounds that were played up to the point of interruption. What was revolutionary about Pascal's and Fermat's solution was their forward-looking nature. Instead of considering what happened, they focused on what would happen on average if the game were to be continued. In doing so, they computed the first recorded expectation (Hammond, 2003, pp. 43-45). In an era where events were deemed irrevocably and necessarily predetermined (Rehlinghaus, 2019), this discovery marked a watershed moment in the history of ideas. In the 60 years following Pascal and Fermat's crucial discovery, expected value maximization became the gold standard for rationality until Nikolaus Bernoulli found a counterexample where it yielded a very unreasonable prediction: In 1711, fellow mathematician Pierre Rémond de Montmort challenged Bernoulli about some strategy of play in the card game Le Her (Bellhouse and Fillion, 2015). Bernoulli, struggling to come up with solutions, in turn proposed a series of problems, one of which would bring forth the perhaps most important concept in decision theory (Szpiro, 2020, p. 5). In what became known as the St. Petersburg paradox [after the place of work of Daniel Bernoulli who carried off the laurels of the paradox's resolution], a participant is offered a lottery where a coin is tossed until heads comes up. Every time the head does not come up, the subject's payoff is doubled. With a fair coin the expectation is

$$E(x) = \frac{1}{2} \times 2 + \frac{1}{4} \times 4 + \frac{1}{8} \times 8 + \dots = \sum_{j=1}^{\infty} \frac{1}{2^j} \times 2^j \quad (1)$$

$= 1 + 1 + \dots \to \infty$ (Kolmar, 2022, p. 216).

The expectation of this lottery amounts to infinity, so how much should the subject be willing to pay to participate in this lottery? According to Pascal and Fermat, the subject should be keen on gambling all her money since the expectation is infinite. However, it is obvious that no sane person would do that—the probability of making it beyond toss number five is a meagre 1.6%. Although Daniel Bernoulli is usually credited with the resolution of the St. Petersburg paradox and the creation of expected utility, Gabriel Cramer had already arrived at expected utility's central insight ten years earlier, in 1728. Cramer acknowledged that "there is no person of good sense would wish to give [even just] 20 coins" to play the game. He attributed the paradox to a difference between the mathematical computation and the value of money, which he called the "the common value." "[...] mathematicians value money in proportion to its quantity, and men of good sense in proportion to the usage they may make of it." (Cramer, 1728). From this insight Cramer inferred that additional money must yield more utility to a poor than to a rich man and proposed the square root, which exhibits diminishing marginal increases, as a measure of the utility of money. It thus follows that the expected value and the maximum amount one

[1] For instance, each player contributes 50 Louis d'or and the game is set to last seven rounds. Sadly, the game is cut short after four rounds, with players 1 and 2 having won three and one round, respectively. A previous solution to the problem contended that players 1 and 2 should receive 3/4 and 1/2 respectively, since player 1 won three of the four rounds that had been played. This solution distributes the pot according to the history of the game and is therefore a backward-looking solution. Pascal and Fermat, however, argued that the fair split consisted of awarding 7/8 to player 1 and 1/8 to player 2. For player 1 to win, player 2 must not win more than two of the remaining three rounds. Otherwise, player 2 would end up with four wins overtaking player 1. The respective probability is $1/2 * 1/2 * 1/2 = 1/8$. Hence, the probability that player 1 wins is 1-1/8 = 7/8. (Kay and King, 2020, p. 59)

should be willing to gamble is:

$$E(x) = \frac{1}{2} \times \sqrt{2} + \frac{1}{4} \times \sqrt{4} + \frac{1}{8} \times \sqrt{8} + ... \\ = 2.91 \,\text{(Szpiro, 2020, p. 13).} \tag{2}$$

Daniel Bernoulli used a slightly more substantiated functional form for the utility function, namely the natural logarithm, the use of which yields a similarly satisfying result regarding the maximum permissible wager:

$$E(x) = \frac{1}{2} \times \ln(2) + \frac{1}{4} \times \ln(4) + \frac{1}{8} \times \ln 8 \\ + ... = 4.01 \text{(own computation).} \tag{3}$$

He backed up his choice of functional form by arguing that the additional utility gained from a marginal increase in wealth is inversely proportional to current wealth and then taking the integral.

$$dU(W) = \frac{c dW}{W}, \tag{4}$$

where c can account for varying degrees of risk aversion between individuals. It follows that,

$$\frac{dU(W)}{dW} = \frac{c}{W}. \tag{5}$$

Integrating this expression results in,

$$U(W) = c \ln W + \text{constant.} \\ \text{(Szpiro, 2020, p. 18)} \tag{6}$$

While this line of reasoning constituted an improvement upon Cramer's solution, one must accept that additional utility is inversely proportional to current wealth without further substantiation; it remains unclear whether the argument of inverse proportionality holds. Nonetheless, the resolution of the St. Petersburg Paradox constitutes the first break with a well-established notion of rationality in decision theory. Unlike other breaks, however, this one was universally accepted.

1.1 Axiomatizing Expected Utility with Expected Utility Theory

The next breakthrough took place in 1944, 206 years after Daniel Bernoulli's treatise, when John von Neumann and Oskar Morgenstern laid out Expected Utility Theory (EUT) which was based on three general fundamental axioms[2].

"**Rationality:** the weak preference relation $\geq$ over the set of possible lotteries **L** is complete and transitive.

1. For all $\mathcal{L}_i, \mathcal{L}_j \in$ **L** either $\mathcal{L}_i \mathcal{L}_j$ or $\mathcal{L}_j \mathcal{L}_i$ or both.
2. For all $\mathcal{L}_i, \mathcal{L}_j, \mathcal{L}_k \in$ **L** if $\mathcal{L}_i \mathcal{L}_j$ and $\mathcal{L}_j \mathcal{L}_k$ then $\mathcal{L}_i \mathcal{L}_k$." (Kolmar, 2022, p. 220)

The first part of rationality is called completeness or commensurability and posits that the decision-maker always prefers one lottery over the other or is indifferent to both of them. The second part is called transitivity. Without it preferences would have a cyclical structure, and it has been shown that a decision-maker with intransitive preferences could be taken advantage of to the point where she would be bankrupt (Peterson, 2017, pp. 178-179).

"**Continuity:** Preferences on the set of lotteries **L** are continuous if for all lotteries $\mathcal{L}_i, \mathcal{L}_j, \mathcal{L}_k$ with $\mathcal{L}_i \mathcal{L}_j \mathcal{L}_k$ there is a probability $p \in [0, 1]$ such that $p \otimes \mathcal{L}_i \oplus (1 - p) \otimes \mathcal{L}_k \mathcal{L}_j \wedge \mathcal{L}_j p \otimes \mathcal{L}_i \oplus (1 - p) \otimes \mathcal{L}_k$" (Kolmar, 2022, p. 221).

[2] The axioms have been combined and split in at least three ways; my presentation follows that in (Kolmar, 2022). For alternative axiomatizations see (Hastie and Dawes, 2009; Varian, 1992).

This axiom implies that a decision-maker can always be made indifferent to the option of choosing a lottery that involves the best and the worst outcome and a certain middle outcome. In other words, no outcomes are so desirable or catastrophic that the decision-maker would always choose or avoid a lottery containing them, regardless of the probabilities of these outcomes. Counterexamples include death, loss of a loved one, extreme public humiliation, and other very adverse outcomes. Then again, people often take actions that involve a small, yet real, chance of extreme outcomes. This happens when they cross the street against a red light, drive a car, or take their children on a plane for a vacation. The willingness to risk extremely adverse outcomes has also been found in a health context; HIV patients may be willing to risk death to cure their disease (Kratka et al., 2019).

"**Independence:** Preferences on the set of lotteries are independent if for all lotteries $\mathcal{L}_i, \mathcal{L}_j, \mathcal{L}_k$ with $\mathcal{L}_\rangle \mathcal{L}_|$ and $p \in (0, 1]$ it follows that $p \otimes \mathcal{L}_i \oplus (1 - t) \otimes \mathcal{L}_k p \otimes \mathcal{L}_j \oplus (1 - t) \otimes \mathcal{L}_k$ holds" (Kolmar, 2022, p. 222).

The intuition behind the independence axiom is simple. If a decision-maker has a preference over two lotteries, this preference should not be reversed if a third lottery is added to both lotteries. In other words, if a decision-maker chooses apples over bananas, she should not all of a sudden prefer bananas to apples because she is offered peaches as a third option[3]. If

[3] Legend has it that the logician Sidney Morgenbesser sat in a restaurant and was offered apple or blueberry pie. Right after deciding in favor of the apple pie, the waitress mentioned that cherry pie was also on the menu. Seizing this once in a lifetime opportunity, Morgenbesser said "In that case, I'll have blueberry" (Pinker, 2022)

all three axioms are satisfied, the Von Neumann and Morgenstern Theorem follows:

"**Von Neumann and Morgenstern Theorem:** A preference relation $\geq$ over lotteries $\mathcal{L} \in \mathbf{L}$ satisfies the axioms of rationality, continuity, and independence if and only if there exists a utility function $EU : \mathbf{L} \rightarrow R$ which represents $\geq$ and has the expected-utility property $EU(\mathcal{L}) = \sum_{j=1}^{I} p_j * u(c_j)$ " (Kolmar, 2022, p. 223).

This theorem brings the theory full circle to the notion of expected utility introduced by Daniel Bernoulli 200 years earlier, by deriving the utility function conjured by Bernoulli from a parsimonious set of general rules. What made Von Neumann Morgenstern's EUT so appealing was its axiomatic foundation—resorting to polished conjectures about how additional wealth impacted utility was no longer necessary. Abiding by the axioms implied consistency between choices and the decision-maker's values; a Von Neumann Morgenstern decision-maker makes choices that are consistent with her goals. However, in the decades following Von Neumann and Morgenstern's publication, more and more violations of the axioms were unearthed.

The most shocking violation was uncovered by Nobel laureate Maurice Allais, who found that people—among them elite economists—consistently violated the independence axiom (see Heukelom, 2015; Kolmar, 2022, pp. 224-226); losses and gains are evaluated differently, and people do not treat probabilities objectively [4]. This posed serious problems to EUT regarding its status as a descriptive theory of human behavior.

1.2 Towards a Psychologically Informed Model of Decision-Making with Prospect Theory

In 1979, Daniel Kahneman and Amos Tversky tackled the framing issue when they developed Prospect Theory (PT). PT was established on the back of the manifold documented violations of EUT and is recognized as a superior explanation of behavior compared to EUT (Szpiro, 2020, p. 200). According to Kolmar (2022), it delivered six distinct yet connected main insights that contrast with EUT (pp. 288-289).

Firstly, utility is no longer measured in absolute terms but relative to a reference point: "An essential feature of the present theory is that the carriers of value are changes in wealth or welfare, rather than final states." (Kahneman and Tversky, 2013, p. 279). The reference point represents what an individual perceives as standard or appropriate in a given situation. This could be the status quo but also a deviation from the status quo which the decision-maker considers normal. Lotteries in which the decision-maker stands only to gain vis-à-vis the reference point cause the decision-maker to choose in accordance with the standard notion of risk-aversion. In situations where she can only lose relative to her reference point, she becomes risk-loving (Kolmar, 2022, pp. 288-289).

Secondly, a decision-maker conducts an initial examination of the decision problem at hand. During this process, she develops a mental model of the problem, the formation of which is partly deliberate and conscious, partly automatic and unconscious. For instance, the reference point forms an important component of the mental model. Generally, the mental model differs from the actual decision problem, and may vary by person so that an objective reference point does not exist, for example. The purpose of the mental model is complexity reduction; by molding the decision problem into something intelligible and tractable, the problem becomes manageable. Numerous variables that are not necessarily central to the decision problem enter and shape the mental model: It is a function of the decision-maker's past experiences, societal norms, external protocols, cultural expectations, etc (Kolmar, 2022, p. 289).

Thirdly, decision-making under risk is conceptualized as a two-stage process, the editing, and the evaluation stage. The former concerns the formation of the

[4] In addition, people do not always abide by the completeness maxim; they are not always willing to choose among options if they touch upon certain things considered sacrosanct such as the value of a human life, an endangered species, organs, the environment, health care, or the arts (Pinker, 2022). What is more, from a revealed preference approach, the inability to choose does not necessarily imply indifference. Consider a US citizen who refuses to vote for either the Democratic or the Republican candidate, a jobseeker who has received two offers and cannot make up his mind which offer to accept, or a bachelorette who is being wooed by two men and struggles to make a decision. The citizen's refusal to vote could indicate a general weariness of politics rather than both candidates being equally desirable in the citizen's eyes. Likewise, indecision regarding potential mates or jobs does not automatically follow from indifference, but could hint at a psychological inclination or even a disorder such as aboulomania. Besides the state of indifference, not all choices reflect a genuine (weak) preference: Social pressure and conformity, impulsivity and lack of self-control, misjudgment and cognitive bias, and decision fatigue might all lead to choices that are at odds with one's preference. Economists should thus tread with care when using the revealed preference paradigm to infer preference from behavior.

mental model and consists of six elements. The first element is termed coding and pertains to the reference dependence and the irrelevance of final states for utility. The second element is called combination and postulates that a decision-maker sums the probabilities associated with identical outcomes, e.g., a 10% probability of losing $10 and a 40% probability of losing $10 will be combined to a 50% probability of losing $50. The third element is named segregation, whereby a decision-maker segregates riskless/certain outcomes from risky prospects. This implies that a lottery whose two outcomes occur with equal probability is split into a sure prospect and a risky one. For example, if the decision-maker faces a 50% probability of gaining $50 and a 50% probability of obtaining $100, she mentally represents this lottery as a 100% probability of winning $50 and a 50% probability of receiving an additional $50. The fourth element is referred to as cancellation and concerns the decision-maker ignoring outcomes that materialize with the same probability when comparing prospects. Suppose a decision-maker has to choose among the following lotteries:

(A) 70% probability of losing $80; 15% probability of losing $10; 15% probability of losing $0.

(B) 70% probability of losing $80; 3% probability of losing $50; 27% probability of losing $0.

She cancels the identical components and is left with the simplified gambles:

(*A) 15% probability of losing $10; 15% probability of losing $0.

(*B) 70% probability of losing $80; 3% probability of losing $50; 27% probability of losing $0.

The fourth element is termed simplification and deals with the rounding not only of probabilities but also of outcomes. The decision-maker represents probabilities hovering around 50% as 50% and rounds very low or very high probabilities to zero and 100%, respectively. In addition, she normalizes (unnecessarily) complicated outcomes to some easier anchor, e.g., she mentally depicts an outcome of $1,002.736 as $1,000. The sixth element is labeled detection of dominance and involves the decision-maker rejecting out of hand options that are worse in terms of both probabilities as well as outcomes compared to the alternatives. Once the decision-maker has completed the six editing operations thus forming her mental model, she advances to the evaluation phase where she decides based on this model (Kolmar, 2022, p. 289; Szpiro, 2020, pp. 200-201).

Fifthly, the disutility from a loss is greater than the utility from an equivalent gain (Kolmar, 2022, p. 289). In other words, "losses loom larger than gains" (Kahneman and Tversky, 2013, p. 279), a phenomenon dubbed loss aversion. Empirically, people need to be compensated for the chance of incurring a loss with the equally chanced prospect of making a gain that is 1.5 to 2.5 times the potential loss (Kahneman and Tversky, 2013, p. 284).

Sixthly, a decision-maker does not treat probabilities objectively but systematically distorts them—even if objective probabilities exist (Kolmar, 2022, p. 289). Small probabilities are consistently overweighted, while intermediate to large probabilities are underweighted (Kahneman and Tversky, 2013). [5]

Published in Econometrica, which is one of the most prestigious economics journals, Kahneman and Tverky's 1979 article ranks among the most often cited papers in the social sciences (Kahneman, 2012, p. 271).

1979 did not mark the end of decision theory history. Instead, the field has retained its vibrancy and continues to generate new insights as evidenced by, e.g., Paul Glimcher's Expected Subjective Value Theory which proposes a neuroeconomic framework that incorporates both EUT and PT (Tymula and Glimcher, 2022)[6].

This article has painted a picture of the history of decision theory under risk as a discipline that sprung from human-made games of chance and spiraled into a general theory of decision-making under risk. This contextualization enables decision theorists and non-decision theorists alike to make sense of an otherwise obscure and unintuitive field. As the towering John von Neumann (1949) himself put it:

> "There is no sense being precise when you don't even know what you're talking about."

[5] Kahneman and Tversky were not the first ones to consider probability distortion; Nicolaus Bernoulli in 1728 proposed that subjects in the St. Petersburg Paradox completely disregard probabilities below 1/32 by setting them to zero (Bernoulli, 1728)

[6] Reading the paper is trivial and is left as an exercise to the reader.

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